Modelling nonlinear Schrödinger superfluid turbulence

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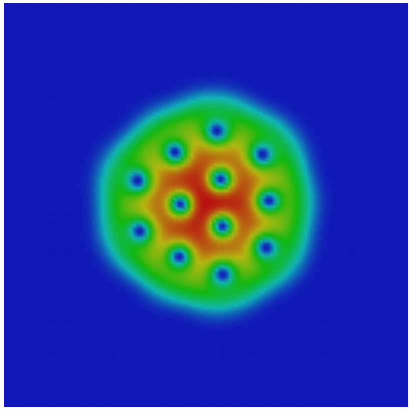
July 7, 2018 : The AIMS Conference Series on Dynamical Systems and Differential Equations, “Advances in mathematical modelling and numerical simulation of superfluid”

Plan of talk

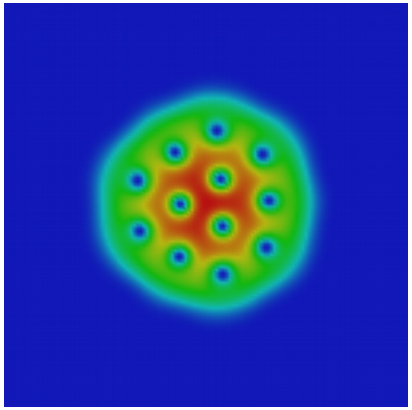
- Can vortices enter a trapped condensate under rotation with the conserving system?



- Transition between superfluid turbulence and steady superflow

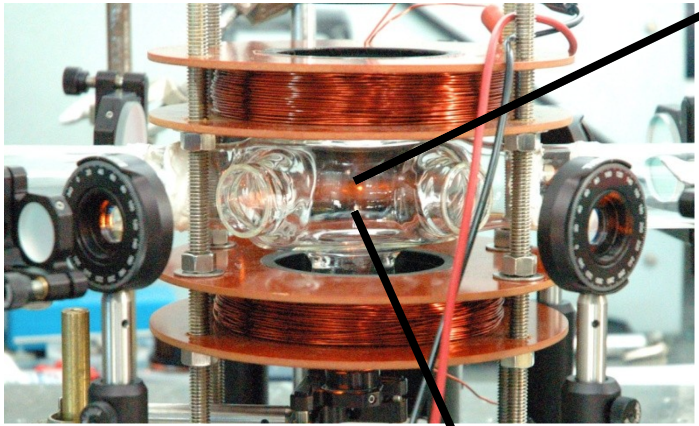
Plan of talk

- Can vortices enter a trapped condensate under rotation with the conserving system?



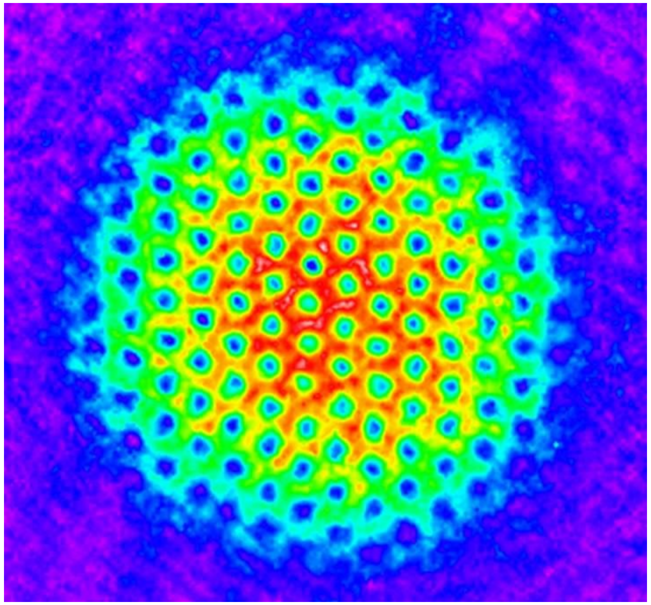
- Transition between superfluid turbulence and steady superflow

Vortex entering dynamics in ultracold atoms

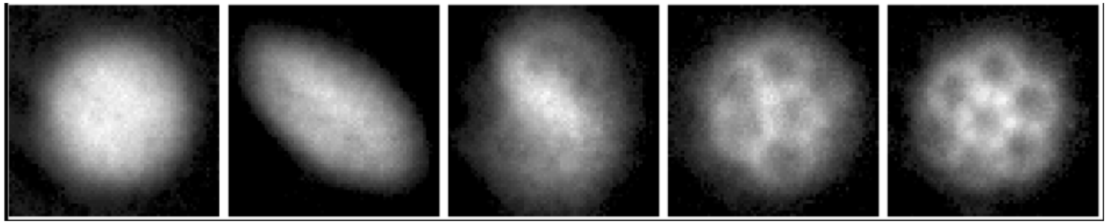


Ultracold atoms in a chamber

Triangular vortex lattice under the rotation



Vortex entering dynamics



Mathematical modelling: Nonlinear Schrodinger equation

$$i\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + V(x,y) + g|\psi|^2 + \Omega L\right]\psi$$

kinetic energy — trapping potential — two-body — rotation

$$V(x,y) = \frac{1}{2}\left\{(1+\epsilon)^2 x^2 + y^2\right\}$$: weakly elliptic harmonic trap

$$L = i\left(x\partial_y - y\partial_x\right)$$: angular momentum operator

Extension to energy-dissipating system with γ

$$(i-\gamma)\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + V(x,y) + g|\psi|^2 + \Omega L - \mu\right]\psi$$

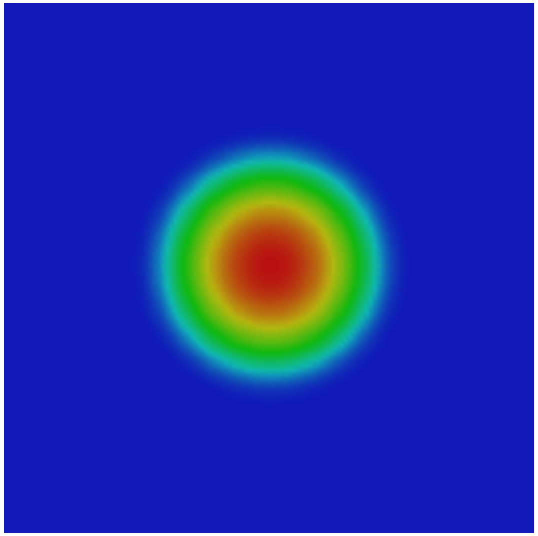
$\mu = \mu(t)$: Lagrange multiplier for $\int d^2x\,|\psi|^2 = 1$

Vortex entering dynamics

$$(i-\gamma)\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)+V(x,y)+g|\psi|^2+\Omega L-\mu\right]\psi$$

μ : Lagrange multiplier for $\int d^2x\,|\psi|^2 = 1$

Initial state : ground state for $\mu\psi = \left[-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)+V(x,y)+g|\psi|^2\right]\psi$



$g = 500$

$\Omega = 0.75$

$\epsilon = 0.05$

$\gamma = 0.02$

Vortex entering dynamics

$$(i-\gamma)\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)+V(x,y)+g|\psi|^2+\Omega L-\mu\right]\psi$$

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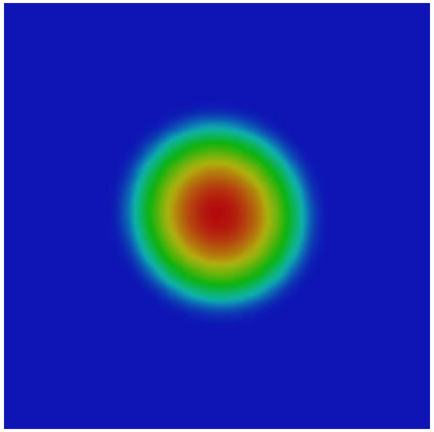


$g = 500$

$\Omega = 0.75$

$\epsilon = 0.05$

$\gamma = 0.02$

What is the dissipation γ

$$(i-\gamma)\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)+V(x,y)+g|\psi|^2+\Omega L-\mu\right]\psi$$

$$\mathcal{E} = \int d^2x \left[\frac{1}{2}\left(|\partial_x\psi|^2+|\partial_y\psi|^2\right)+\left\{V(x,y)+\frac{g}{2}|\psi|^2\right\}|\psi|^2+\Omega \mathrm{Re}\left[\psi^* L\psi\right]\right]$$

$$\mathcal{E}_{\text{no vortex}} > \mathcal{E}_{\text{vortex lattice}}$$

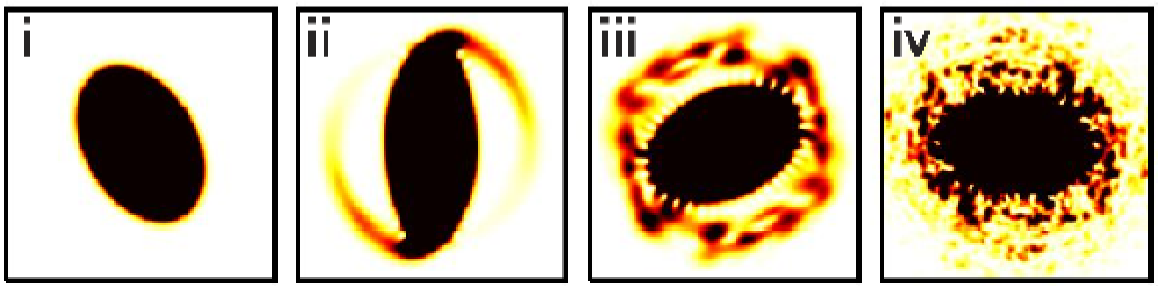
It is reasonable to introduce γ to obtain the vortex lattice state with $\mathcal{E}_{\text{vortex lattice}}$.

Some physicists dislike γ because its physical explanation and quantitative estimation are difficult (other terms can be easily introduced).

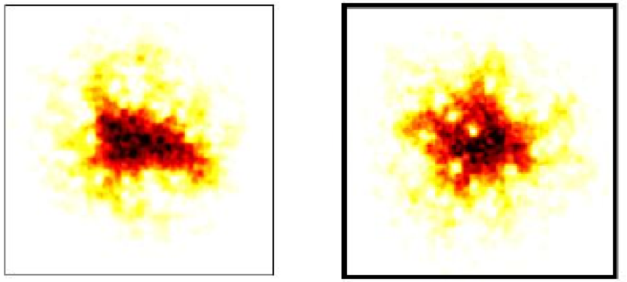
What happens without γ

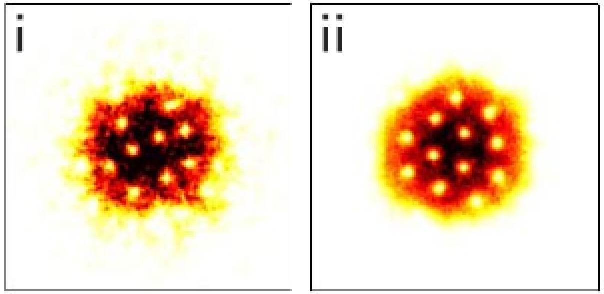
$$i\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + V(x,y) + g|\psi|^2 + \Omega L\right]\psi$$

N. G. Parker and C. S. Adams, Phys. Rev. Lett **95**, 145301 (2005).



turbulence?





- Vortices enter the condensate and finally form the lattice even without the dissipation γ
- Some physicists believe that the dissipation is not need to interpret the lattice formation dynamics

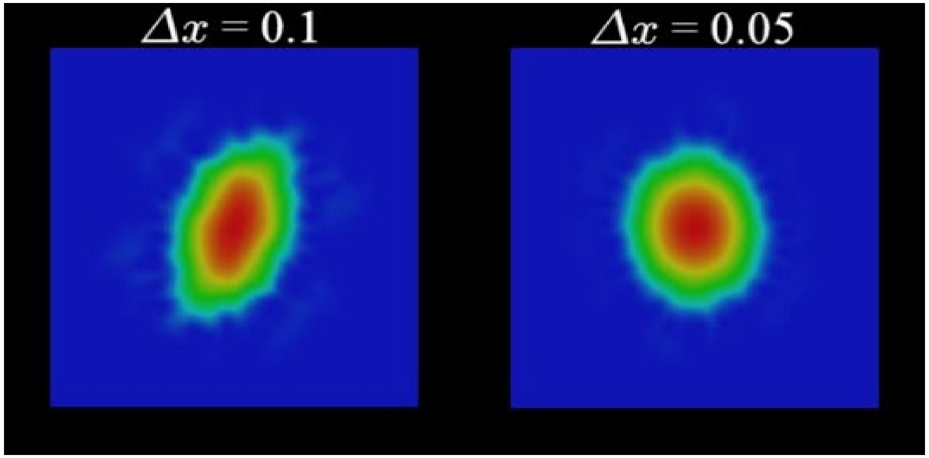
Is vortex entering dynamics true?

$$i\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + V(x,y) + g|\psi|^2 + \Omega L\right]\psi$$

$$V(x,y) = \frac{1}{2}\left\{(1+\epsilon)^2 x^2 + y^2\right\}$$

$$L = i\left(x\partial_y - y\partial_x\right)$$

System size : $X = 12.5$



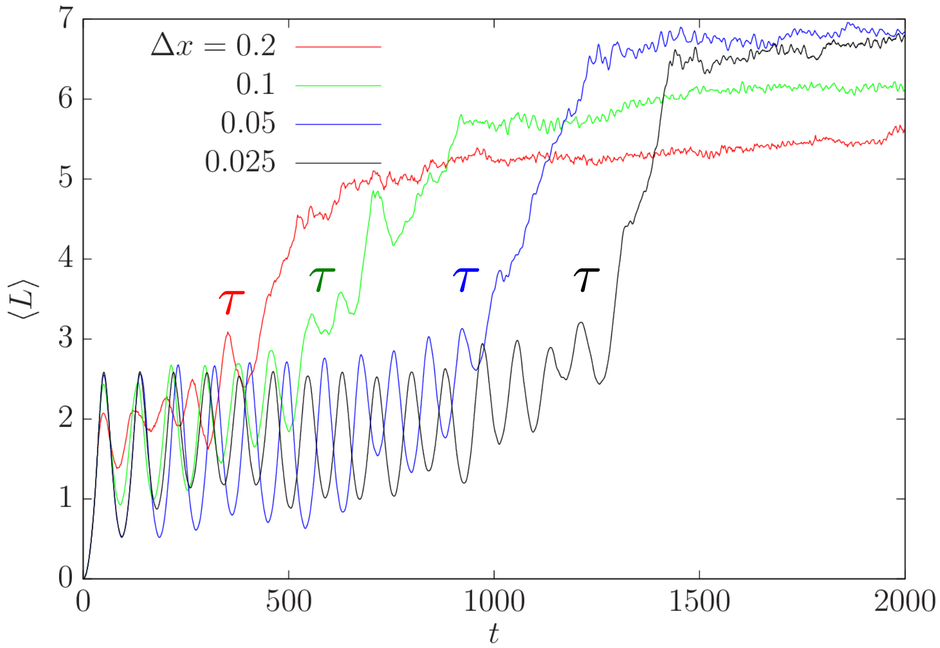
$g = 500 \quad \Omega = 0.75 \quad \epsilon = 0.05$

space : finite difference

time : Crank-Nicolson

Angular momentum

$$\langle L \rangle = \int d^2x \, \mathrm{Re}[\psi^* L \psi]$$

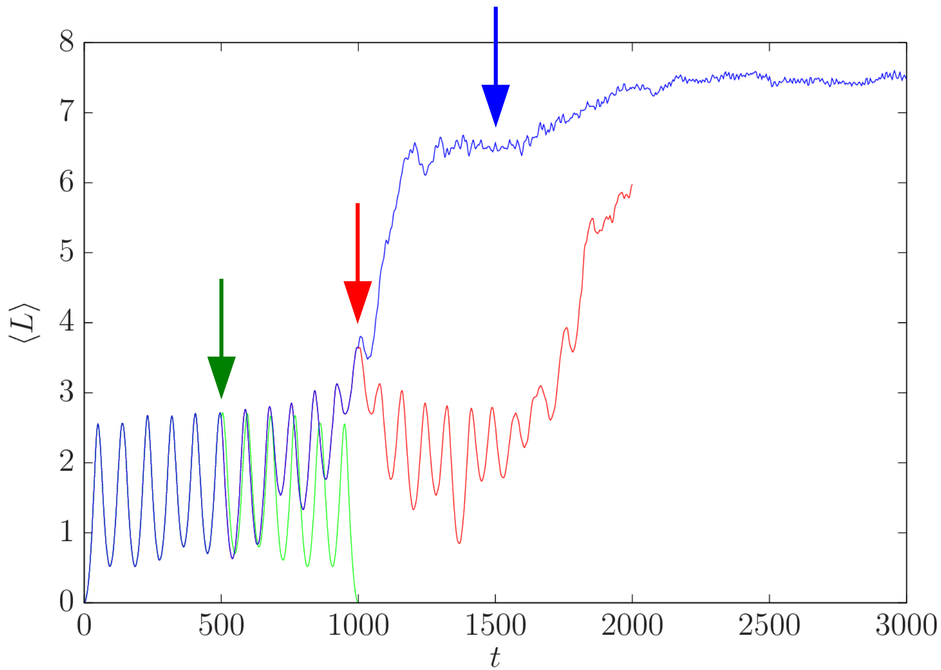


- Dynamics strongly depends on the numerical accuracy.
- When vortices enter the condensate, $\langle L \rangle$ suddenly increases
- Vortex entering time τ increases with the numerical accuracy (one expactation : $\tau \to \infty$ with $\Delta x \to 0$).

Time reversal

$$i\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2+\partial_y^2\right)+V(x,y)+g|\psi|^2+\Omega L\right]\psi$$

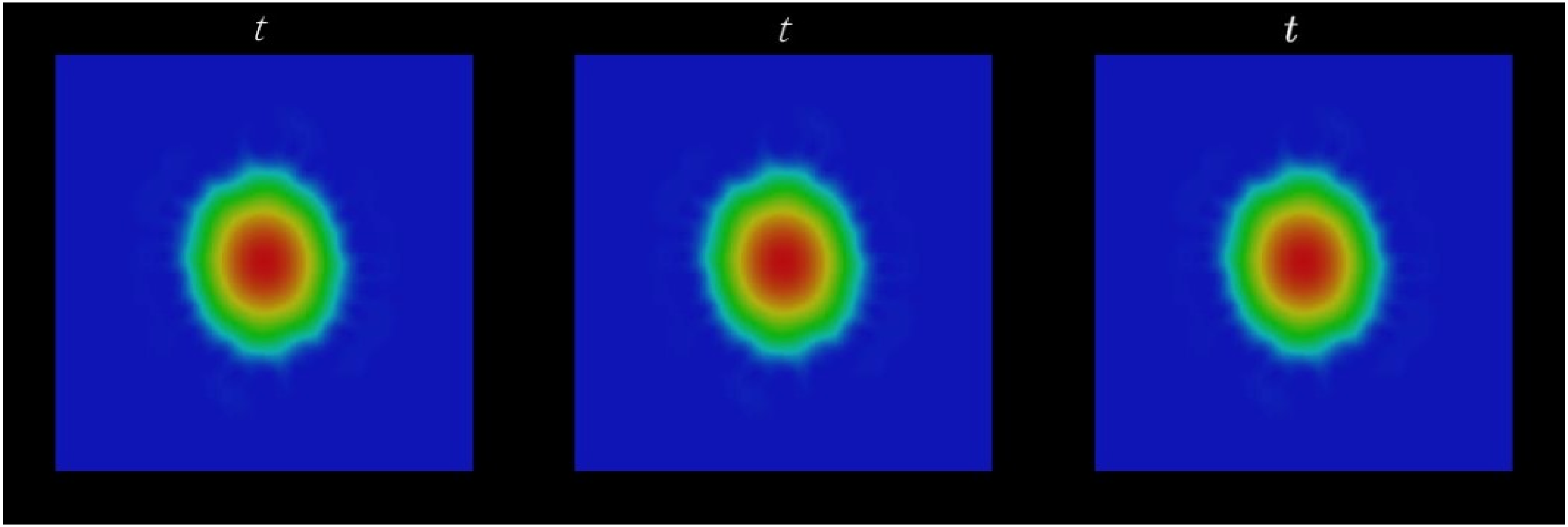
Time-reversal symmetry : $t \to -t \quad \psi \to \psi^* \quad \Omega \to -\Omega$



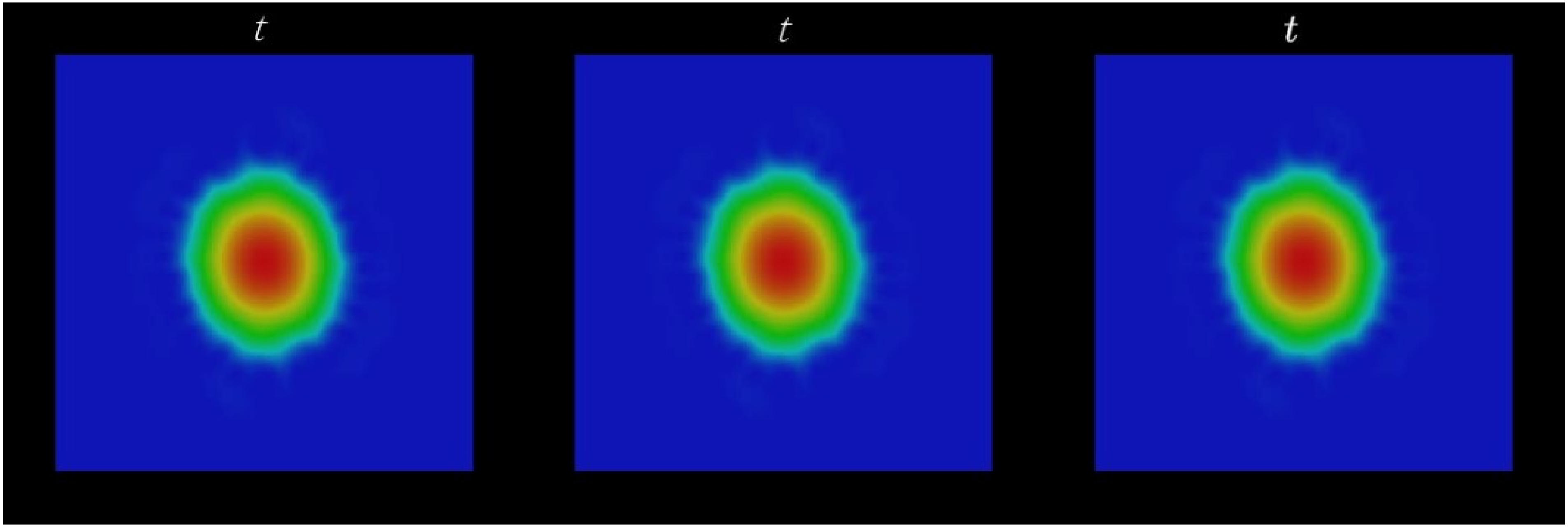
After vortices enter the condensate, time-reversal symmetry is not hold.

Vortex entering time τ : $\int d^2x \; \{|\psi^+(\tau-\Delta t)-\psi^-(\tau+\Delta t)|^2\} \geq \Delta t$

Time reversal

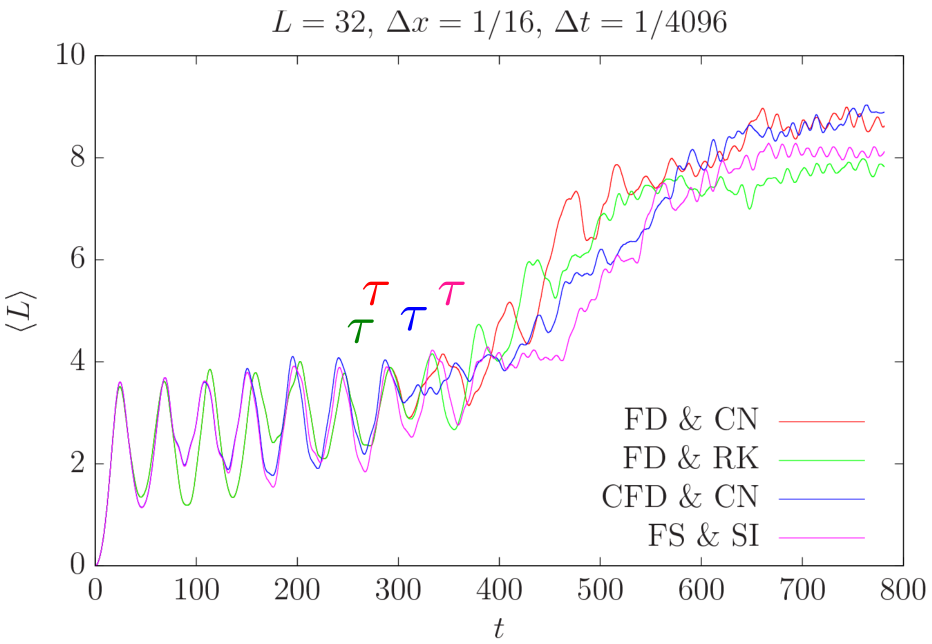


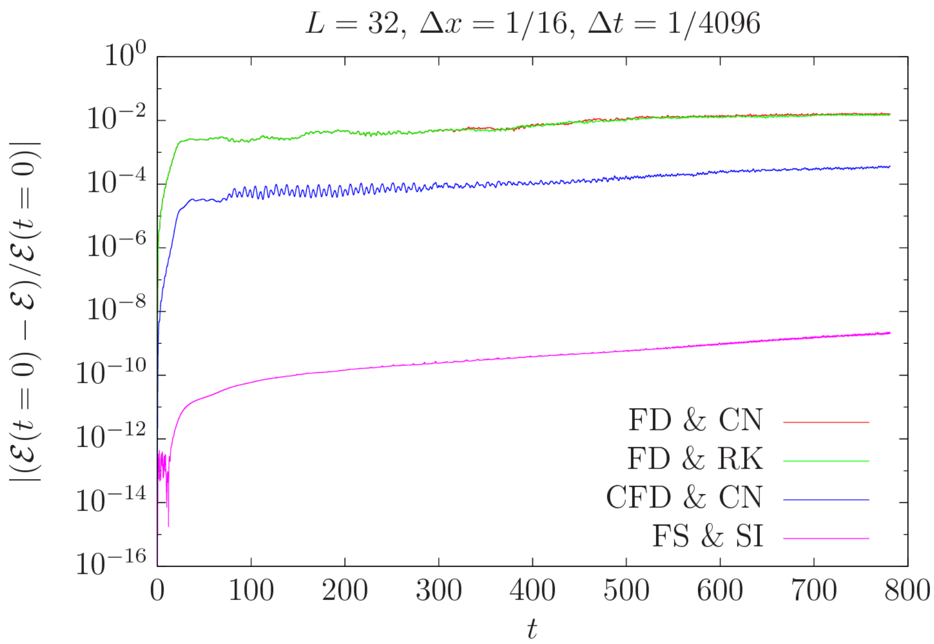
Time reversal



- Time-reversal symmetry is truly broken?
- Or an effect of the positive Lyapunov exponent?

Results also strongly depend on numerical schemes





- FD & CN : Finite difference (space) and Crank-Nicolson (time)
- FD & RK : Finite difference (space) and Runge-Kutta (time)
- CFD & CN : Compact finite difference (space) and Crank-Nicolson (time)
- FS & SI : Fourier spectral (space) and Symplectic integrator (time)

What we want to know

$$i\partial_t\psi = \left[-\frac{1}{2}\left(\partial_x^2 + \partial_y^2\right) + V(x,y) + g|\psi|^2 + \Omega L\right]\psi$$

$$V(x,y) = \frac{1}{2}\left\{(1+\epsilon)^2 x^2 + y^2\right\} \qquad L = i\left(x\partial_y - y\partial_x\right)$$

Vortices truly enter the condensate and form a lattice?

- Are vortex entering dynamics seen in numerical simulations true?
- Does the smooth solution of the equation exist in any t?
- Does the time-reversal symmetry of the equation hold in any t?

We want to know whether we need the dissipation or not to explain the vortex-entering dynamics

Special thanks to D. Ionut

Plan of talk

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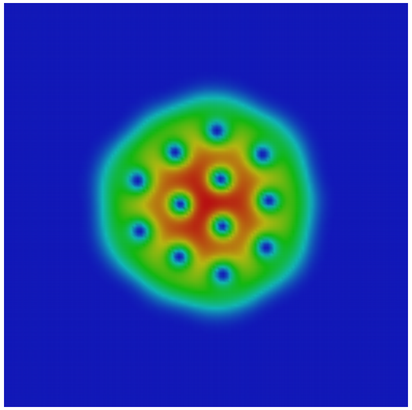
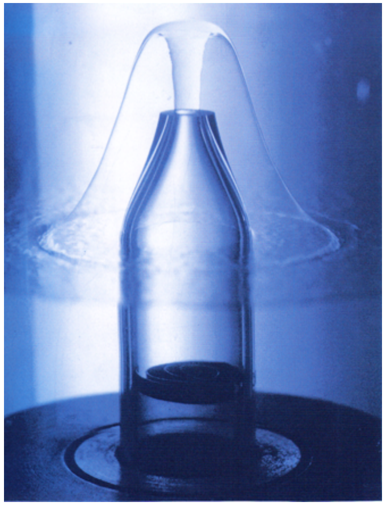
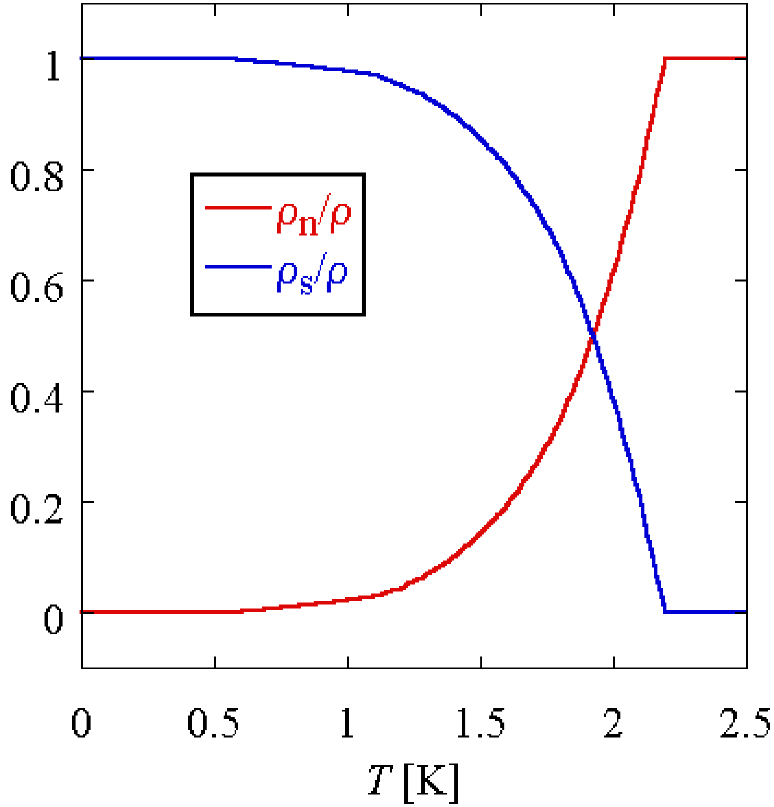


- Transition between superfluid turbulence and steady superflow

Superfluid helium and two-fluid model

Superfluid helium



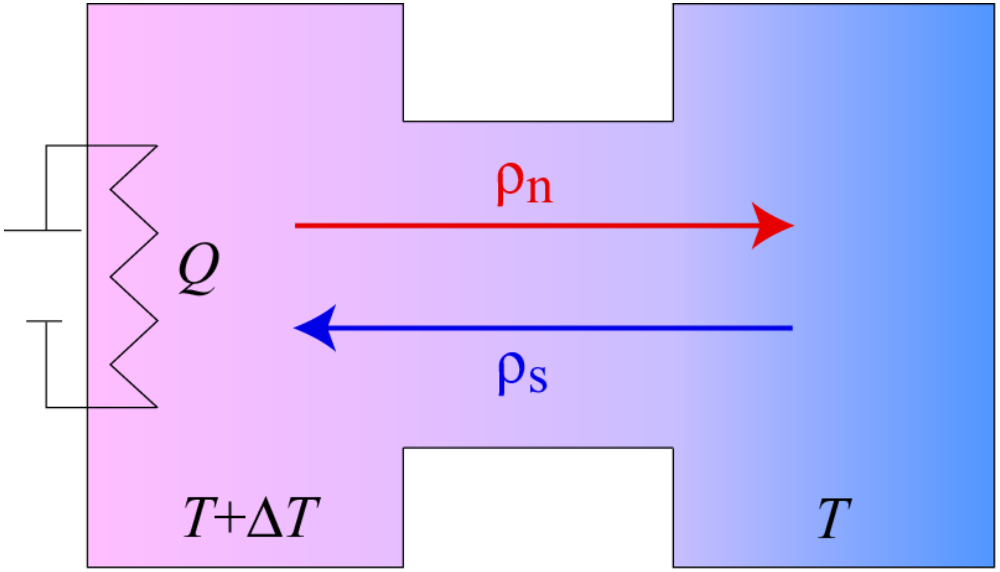


fluid density $\rho = \rho_n + \rho_s$

ρ_n : normal fluid (viscous)

ρ_s : superfluid (inviscid)

Thermal counterflow



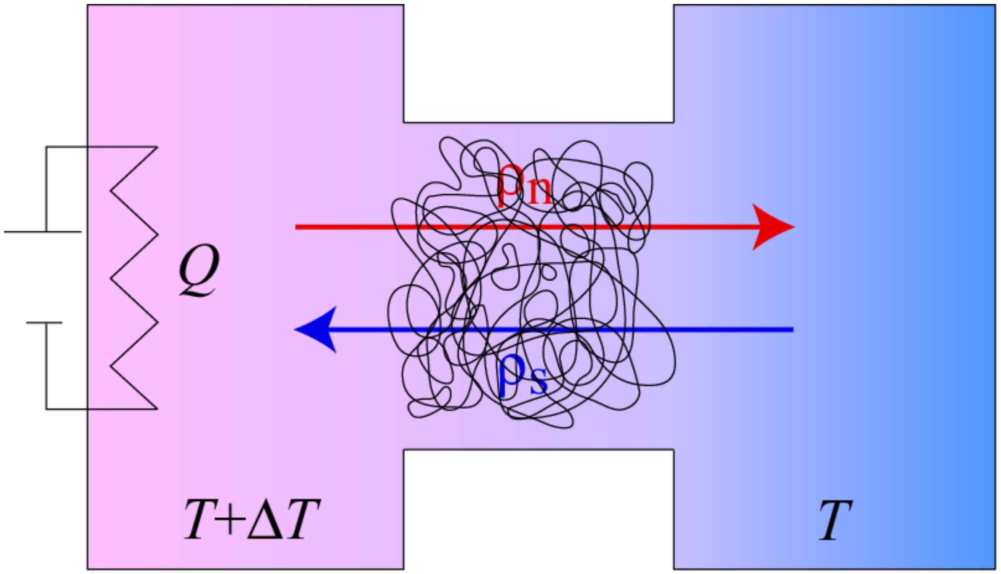
Counterflow : $\rho_s \boldsymbol{v}_s = -\rho_n \boldsymbol{v}_n$

Normal flow : hot $\longrightarrow$ cold

Superfluid : cold $\longrightarrow$ hot

Heating a side of a pipe $\longrightarrow$ Counterflow wipe the temperature difference

Thermal counterflow



Counterflow : $\rho_s \boldsymbol{v}_s = -\rho_n \boldsymbol{v}_n$

Normal flow : hot $\longrightarrow$ cold
Superfluid : cold $\longrightarrow$ hot

Transition to superfluid turbulence with vortices occurs above the critical counterflow velocity

We discuss this phenomenon in the nonlinear Schrödinger framework

Hydrodynamic equations for superfluid helium

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial r_j}\left(\Pi_{ij} + \Pi'_{ij}\right) = 0 \qquad \frac{\partial \boldsymbol{v}_\mathrm{s}}{\partial t} + \nabla\left(\mu + \frac{1}{2}v_\mathrm{s}^2 + h'\right) = 0$$

$$\rho = \rho_\mathrm{n} + \rho_\mathrm{s} \qquad \boldsymbol{j} = \rho_\mathrm{n}\boldsymbol{v}_\mathrm{n} + \rho_\mathrm{s}\boldsymbol{v}_\mathrm{s} \qquad \boldsymbol{w} = \boldsymbol{v}_\mathrm{n} - \boldsymbol{v}_\mathrm{s}$$

$$\Pi_{ij} = P\delta_{ij} + \rho_\mathrm{s} v_{\mathrm{s}i} v_{\mathrm{s}j} + \rho_\mathrm{n} v_{\mathrm{n}i} v_{\mathrm{n}j}$$

$$\Pi'_{ij} = -\eta_\mathrm{n}\left(\frac{\partial v_{\mathrm{n}i}}{\partial r_j} + \frac{\partial v_{\mathrm{n}j}}{\partial r_i} - \frac{2}{3}\delta_{ij}\nabla \cdot \boldsymbol{v}_\mathrm{n}\right) + \delta_{ij}\left\{\zeta_1 \nabla \cdot (\rho_s \boldsymbol{w}) - \zeta_2 \nabla \cdot \boldsymbol{v}_\mathrm{n}\right\}$$

$$h' = \zeta_3 \nabla \cdot (\rho_\mathrm{s}\boldsymbol{w}) - \zeta_1 \nabla \cdot \boldsymbol{v}_\mathrm{n}$$

Hydrodynamic equations for superfluid helium

Simple situation : constant $\rho_{\rm n}$ and $\boldsymbol{v}_{\rm n}$

$$\frac{\partial \rho_{\rm s}}{\partial t} + \nabla \cdot (\rho_{\rm s} \boldsymbol{v}_{\rm s}) = 0$$

$$\frac{\partial \boldsymbol{v}_{\rm s}}{\partial t} + \nabla \left[\mu + \frac{1}{2} v_{\rm s}^2 + \zeta_3 \nabla \cdot \{ \rho_{\rm s} (\boldsymbol{v}_{\rm n} - \boldsymbol{v}_{\rm s}) \} - \zeta_1 \nabla \cdot \boldsymbol{v}_{\rm n} \right] = 0$$

$$i \frac{\partial \psi}{\partial t} = \left\{ -\frac{1}{2} \nabla^2 + \mu \left(|\psi|^2 \right) \right\} \psi + i\gamma \left(\nabla - \boldsymbol{v}_{\rm n} \right)^2 \psi$$

$$\rho_{\rm s} = |\psi|^2$$

$$\boldsymbol{v}_{\rm s} = \nabla \mathrm{Arg}[\psi]$$

$$\frac{\partial \rho_{\rm s}}{\partial t} + \nabla \cdot (\rho_{\rm s} \boldsymbol{v}_{\rm s}) = 0$$

$$\frac{\partial \boldsymbol{v}_{\rm s}}{\partial t} + \nabla \left[U + \frac{1}{2} \boldsymbol{v}_{\rm s}^2 + \frac{\gamma}{\rho_{\rm s}} \nabla \cdot \{ \rho_{\rm s} (\boldsymbol{v}_{\rm n} - \boldsymbol{v}_{s}) \} \right] = \frac{1}{2\rho_{\rm s}} \frac{\nabla^2 \sqrt{\rho_{\rm s}}}{\sqrt{\rho_{\rm s}}}$$

Simulation for thermal counterflow

$$i\frac{\partial\psi}{\partial t} = \left\{-\frac{1}{2}\nabla^2 + \mu\left(|\psi|^2\right)\right\}\psi + i\gamma\left(\nabla - \boldsymbol{v}_\mathrm{n}\right)^2\psi$$

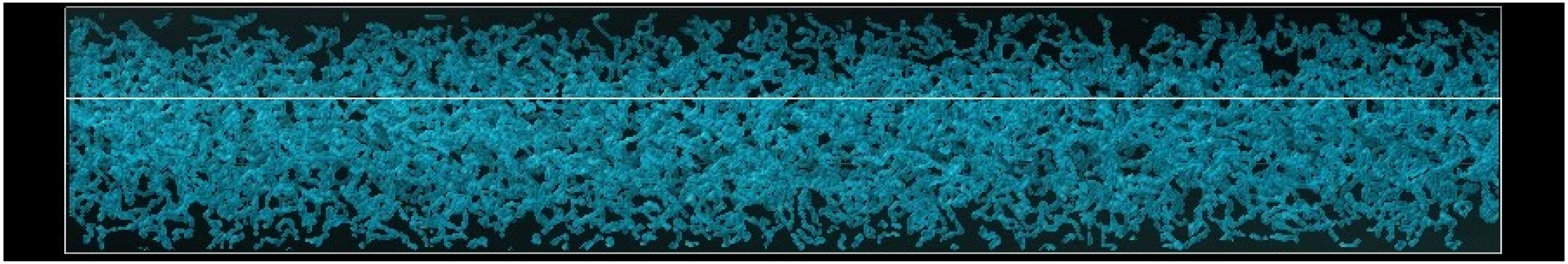
$$\mu = |\psi|^2 - 1$$

Numerical parameter : $\Delta x = 0.5 \quad X = 512 \quad Y = Z = 64$ periodic B. C.

$\boldsymbol{v}_\mathrm{n} = (v_\mathrm{n}, 0, 0)$

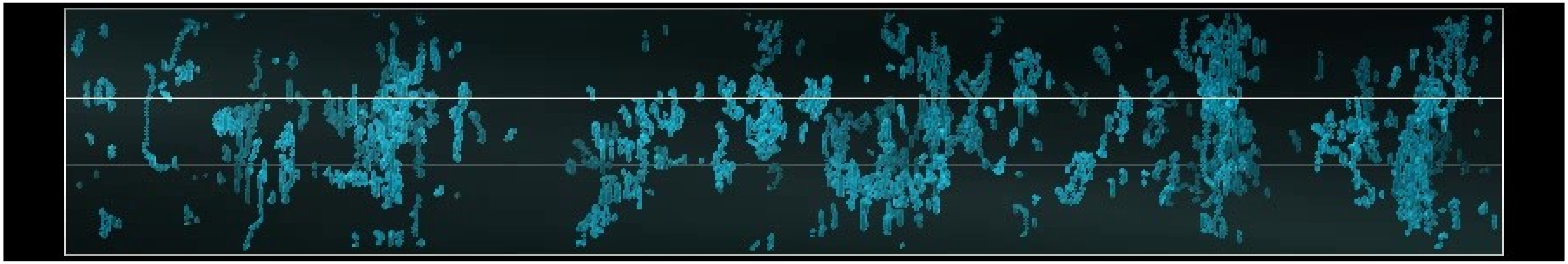
$\psi(t=0) = 1 + 0.05(\xi_1 + i\xi_2) \qquad \langle\xi_i(\boldsymbol{x})\rangle = 0 \quad \langle\xi_i(\boldsymbol{x})\xi_j(\boldsymbol{y}\rangle = \delta_{i,j}\delta(\boldsymbol{x} - \boldsymbol{y})$

From initial state to stationary dynamics with $v_\mathrm{n} = 1$

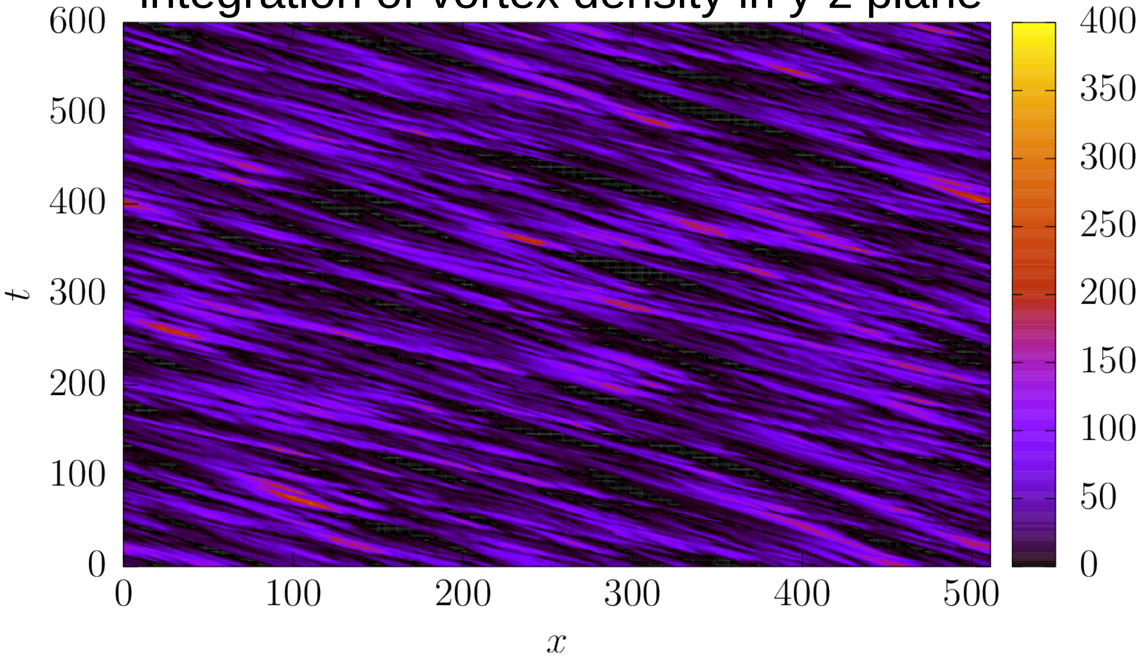


Lowering v_n little by little

$v_n = 0.85$



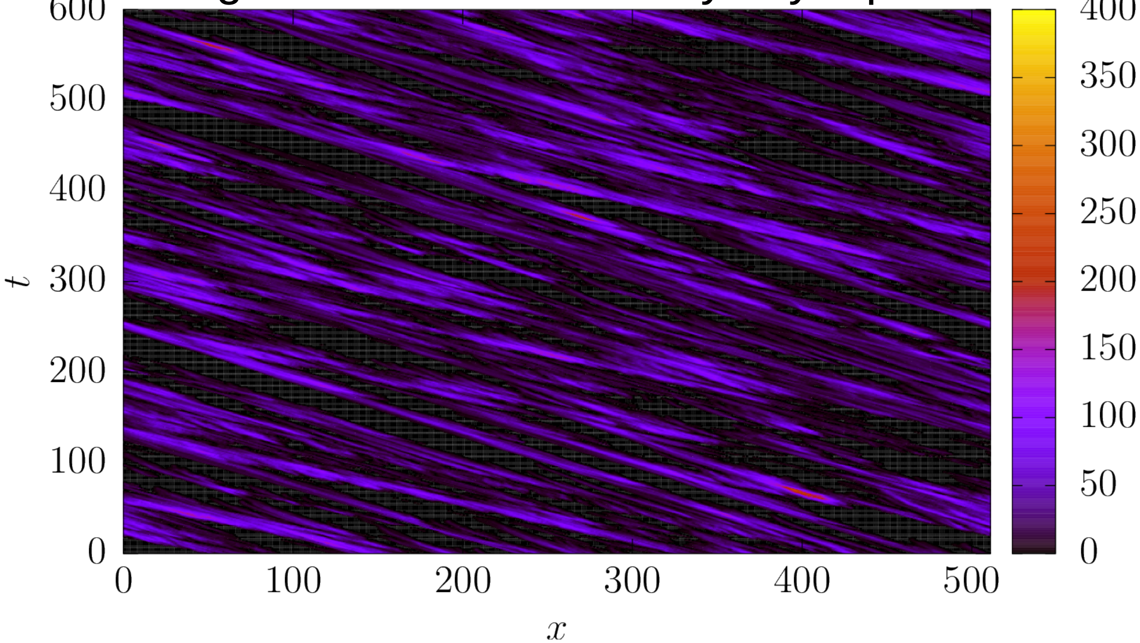
integration of vortex density in y-z plane



Pipe flow turbulence simulation

$$v_n = 0.8$$

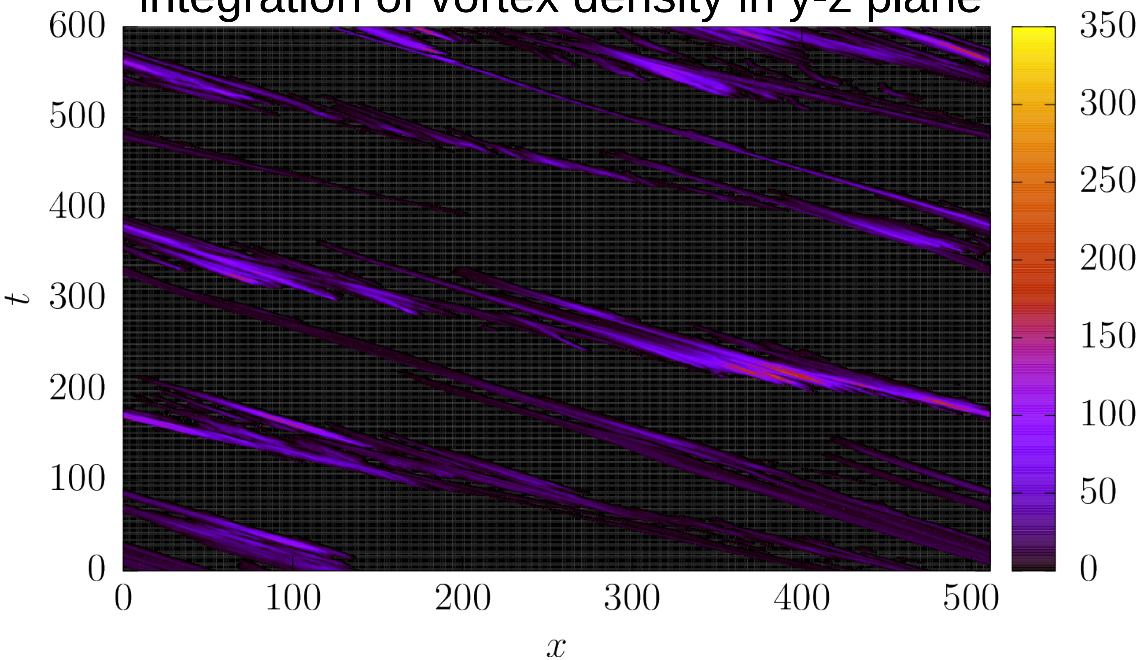
integration of vortex density in y-z plane



Pipe flow turbulence simulation

$v_n = 0.75$

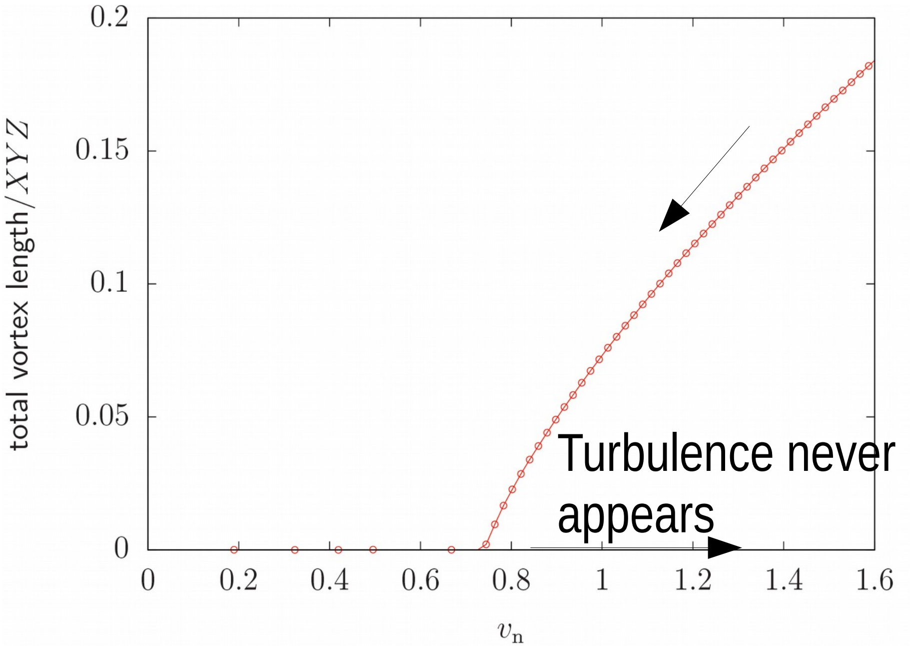
integration of vortex density in y-z plane

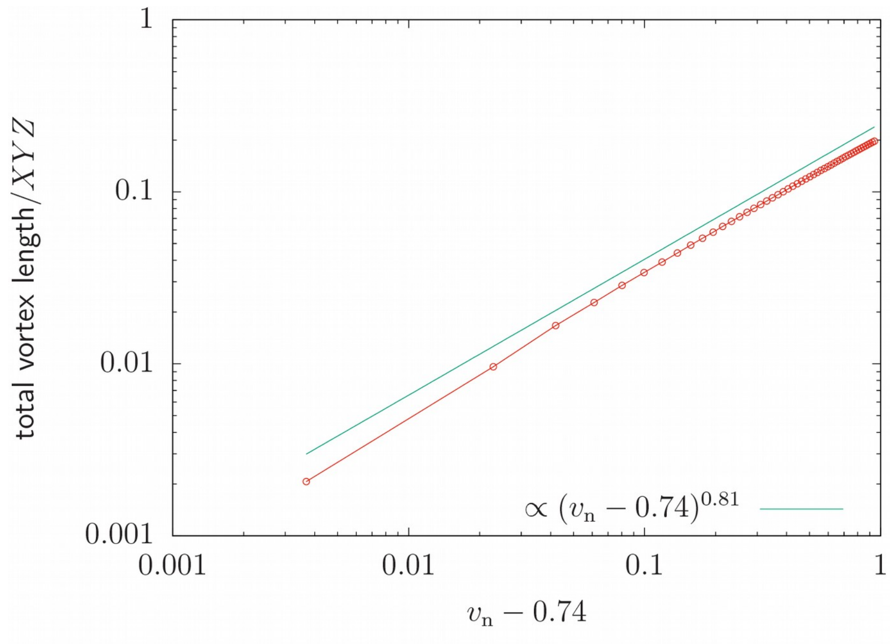


Turbulent is intermittent.

Order parameter of turbulence

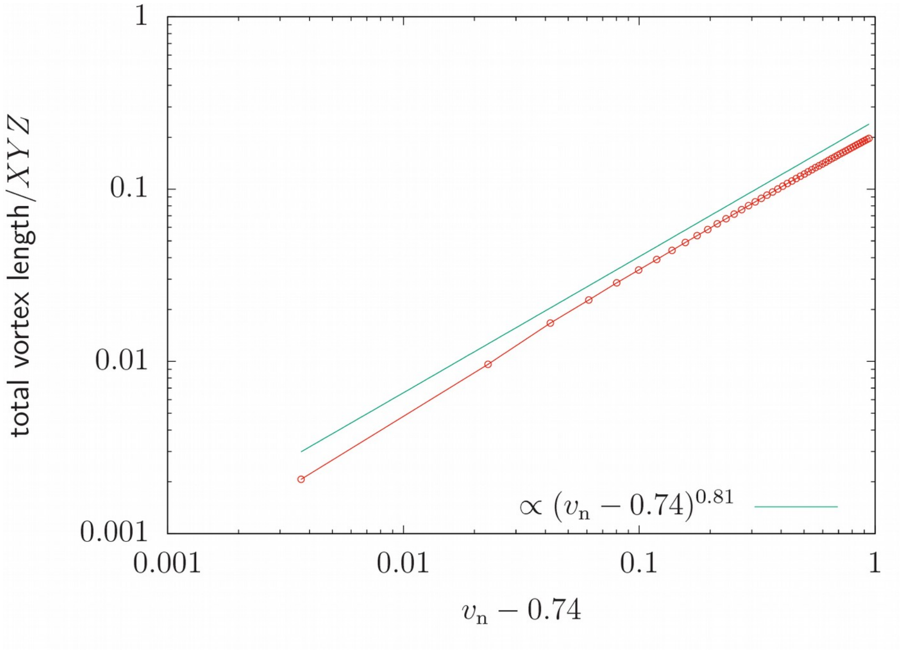
Order parameter of turbulence : total vortex length





Transition looks like second ordered thermodynamic transition in equilibrium (critical exponent : $\beta = 0.81$)

Critical exponent for turbulent transition



critical exponent : $\beta = 0.81$

Same critical exponent can be obtained in a completely different model

$$\frac{\partial \rho}{\partial t} = \rho - b\rho^2 + \nabla^2 \rho + \sqrt{\rho} \cdot \eta \text{ (Itô product)}$$

$$\rho > 0 \qquad \langle \eta(\boldsymbol{x}, t) \rangle = 0 \qquad \langle \eta(\boldsymbol{x}, t)\eta(\boldsymbol{x}', t') \rangle = \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$$

What we want to know

$$i\frac{\partial\psi}{\partial t} = \left\{-\frac{1}{2}\nabla^2 + |\psi|^2 - 1\right\}\psi + i\gamma\left(\nabla - \boldsymbol{v}_{\mathrm{n}}\right)^2\psi$$

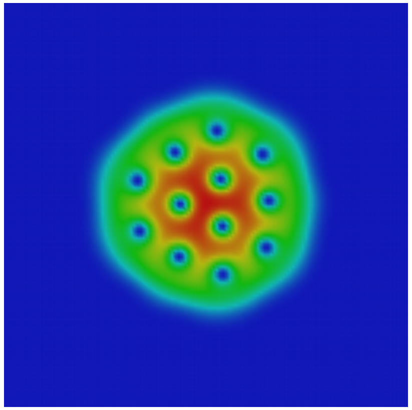
⇕

$$\frac{\partial\rho}{\partial t} = \rho - b\rho^2 + \nabla^2\rho + \sqrt{\rho}\cdot\eta$$

- Is there common mathematical structure? And if so, what is this?
- ρ_{n} and $\boldsymbol{v}_{\mathrm{n}}$ should be time dependent and follow the Navier-Stokes equation with the finite viscosity. How is the turbulent transition modified?

Summary

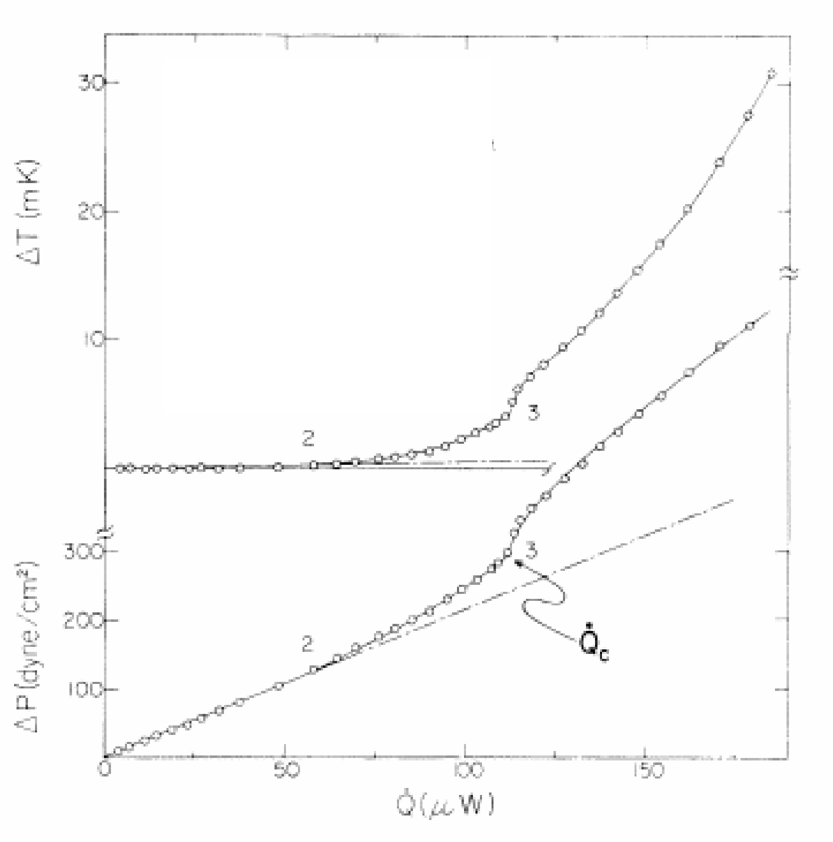
- Do vortices enter the condensate under the conserving dynamics?



- What is the background physics in the turbulent transition?

Experiment？

- Only ^4He experiment（no ^3He）



- Two-step transition
- What is the first transition?
- Second transition is believed to be transition between no vortex and vortices.