

Vector dark-antidark solitary waves in multicomponent Bose-Einstein condensatesI. Danaila,^{1,*} M. A. Khamsehchi,² V. Gokhroo,² P. Engels,² and P. G. Kevrekidis³¹*Laboratoire de Mathématiques Raphaël Salem, Université de Rouen Normandie, 76801 Saint-Étienne-du-Rouvray, France*²*Washington State University, Department of Physics & Astronomy, Pullman, Washington 99164, USA*³*Department of Mathematics and Statistics, University of Massachusetts Amherst, Amherst, Massachusetts 01003-4515, USA*

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Multicomponent Bose-Einstein condensates exhibit an intriguing variety of nonlinear structures. In recent theoretical work [C. Qu, L. P. Pitaevskii, and S. Stringari, *Phys. Rev. Lett.* **116**, 160402 (2016)], the notion of magnetic solitons has been introduced. Here we examine a variant of this concept in the form of vector dark-antidark solitary waves in multicomponent Bose-Einstein condensates (BECs). We first provide concrete experimental evidence for such states in an atomic BEC and subsequently illustrate the broader concept of these states, which are based on the interplay between miscibility and intercomponent repulsion. Armed with this more general conceptual framework, we expand the notion of such states to higher dimensions presenting the possibility of both vortex-antidark states and ring-antidark-ring (dark soliton) states. We perform numerical continuation studies, investigate the existence of these states, and examine their stability using the method of Bogoliubov–de Gennes analysis. Dark-antidark and vortex-antidark states are found to be stable for broad parametric regimes. In the case of ring dark solitons, where the single-component ring state is known to be unstable, the vector entity appears to bear a progressively more and more stabilizing role as the intercomponent coupling is increased.

DOI: [10.1103/PhysRevA.94.053617](https://doi.org/10.1103/PhysRevA.94.053617)**I. INTRODUCTION**

Atomic Bose-Einstein condensates (BECs) offer an excellent test bed for the exploration of waveforms relevant to multicomponent nonlinear wave systems [1,2]. A principal paradigm consists of the dark-bright (DB) solitary wave and related structures such as dark-dark solitary waves that have long been studied theoretically [3–9]. The experimental study of such states was pioneered much earlier in nonlinear optics, including, e.g., the observation of DB solitary wave structures in Refs. [10,11]. Yet, it was the versatility and tunability of BECs that enabled a wide variety of relevant studies initially motivated by the proposal of Ref. [12]. Specifically, the experimental realization of DBs [13] was followed by a string of experiments investigating the dynamics and properties of these features including in-trap oscillations of DBs, their spontaneous generation (e.g., via counterflow experiments), and their interactions both with other DBs and with external potential barriers [14–19].

Very recently, a different type of multicomponent solitons was proposed, the so-called “magnetic solitons” [20]. Given the limited number of experimental investigations in connection to magnetic solitons (in comparison to other soliton families, such as bright or dark solitons) such entities naturally are of considerable interest. The ability to generate them using current state-of-the-art experiments with multicomponent BECs gives them considerable experimental appeal as well. These states have a complementary intensity profile in the two-components ($\psi_1(x,t), \psi_2(x,t)$) and are described by the two-component wave function

$$\begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{pmatrix} = \sqrt{n} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{i\phi_1} \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi_2} \end{pmatrix},$$

where $\theta(x,t)$ characterizes the spatial distribution of the amplitude, n the total density, and ϕ the phase of each component. It is relevant to note that a related idea regarding the formulation of the multicomponent nonlinear wave state was put forth, e.g., in Ref. [21].

A complementary possibility recognized considerably earlier was that of dark-antidark solitary waves [22]. Antidark solitary waves are bright solitary waves on top of a finite background. Here we will avoid calling the structures under investigation “magnetic,” as we do not *a priori* constrain the sum of the densities of the two components to equal that of a single-component ground state, as in the settings of Refs. [20,21]. We will show that the idea of complementary nontrivial components, one of which is antidark, is a very general one and is applicable in several dimensions as well: similar ideas naturally emerge in two dimensions in the form of vortex-antidark and ring-antidark-ring states, which to the best of our understanding have previously not been explored. In particular, a vortex-antidark wave will be a miscible two-dimensional (2D) state where a vortex in the one-component forms a potential well inside which the atomic density of the second component builds up to create an antidark solitary wave. A ring dark soliton, on the other hand, has been extensively studied in one-component BECs [2] as a radial state that starts in the linear limit as a linear combination of the second excited states of an isotropic quantum harmonic oscillator of frequency Ω , i.e., $\sim (\Omega r^2 - 1)e^{-\Omega r^2/2}$ and as the atom number increases it becomes a tanh-shape dark soliton but in the *radial* variable. Hence, again starting from a miscible state that contains a ring dark soliton in the one component, we will observe that this structure will confine atoms of the second component in the form of an antidark ring, creating what we will thus name a ring-antidark-ring state. We motivate and complement our theoretical prediction, numerical verification, and stability analysis of such states

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with an example of an experimental realization in a BEC that confirms that dark-antidark states are straightforward to create and observe the dynamics of in current experimental settings.

The fundamental rationale behind such states is reminiscent of that of the DB entities. For a single-component system, an extensive discussion of the existence and stability of excited states such as dark solitons or vortices can be found in the respective one-dimensional (1D) and 2D chapters of Ref. [2]. In a two-component system with intercomponent repulsion, a dark soliton or a vortex (or a ring) in one component will induce an effective potential, through the intercomponent nonlinearity, on the second component. If now atoms of the second (“bright”) component are added solely within this effective potential, the density suppression in the first component will get filled by atoms of the second component and a DB, a vortex-bright [23,24], or a ring-DB solitary wave [25] will emerge. If, now, the second component features a (spatially extended) ground state profile in the miscible regime, the presence of intercomponent repulsion will produce an effective additional potential which will attract atoms of the second component into the dip of the first one. This generates a bright solitary wave *on top of* the existing nontrivial background, forming an antidark solitary wave. An additional constraint in this case is that the two components need to coexist outside of the dark-antidark (DA), vortex-antidark (VA), or ring-antidark-ring (RAR) structure. This imposes the condition of miscibility between the two species [26–28], i.e., the condition that the intercomponent repulsion should be less than the square root of the product of the intracomponent ones, $0 \leq g_{12} < \sqrt{g_{11}g_{22}}$. We note in passing here that this condition is derived in the context of homogeneous BECs and is only slightly affected by the presence of weak trapping conditions as in the case examples considered herein [29].

Based on the discussion above, there is a straightforward path that one can follow in order to establish such states (at least, numerically) involving an antidark component. One can start at the uncoupled limit of $g_{12} = 0$ with an excited state (e.g., a dark soliton in one dimension, or a vortex or a ring dark soliton in two dimensions) in one component and a fundamental (ground) state in the second component. Then, after turning on the intercomponent coupling, the dip of the excited state in the first component will induce an effective attracting potential (due to the intercomponent repulsion) in the second component, attracting some atoms into the dip while maintaining (due to miscibility) the background of the second component. By construction, an antidark structure will be formed.

Such a state, as we will see in detail below, will continue to exist for values of g_{12} up to the miscibility-immiscibility threshold. To discuss these types of states, we will proceed as follows: in Sec. II we will provide an example of an experimental realization of a dark-antidark solitary wave that will serve as a key motivation for the corresponding theoretical more in-depth study. In Sec. III we will explore the relevant states numerically, using numerical continuation and bifurcation theory, starting from the uncoupled limit described above. Finally, in Sec. IV we will summarize our conclusions and present some intriguing possibilities for future work.

II. EXPERIMENTAL RESULTS

To motivate our discussion, we begin by presenting experimental evidence for the existence, stability, and dynamics of a dark-antidark solitary wave, shown in Fig. 1. In our experiments we observe these features in two-component BECs confined in an elongated dipole trap. The experiments begin by creating a BEC of approximately 0.8×10^6 ^{87}Rb

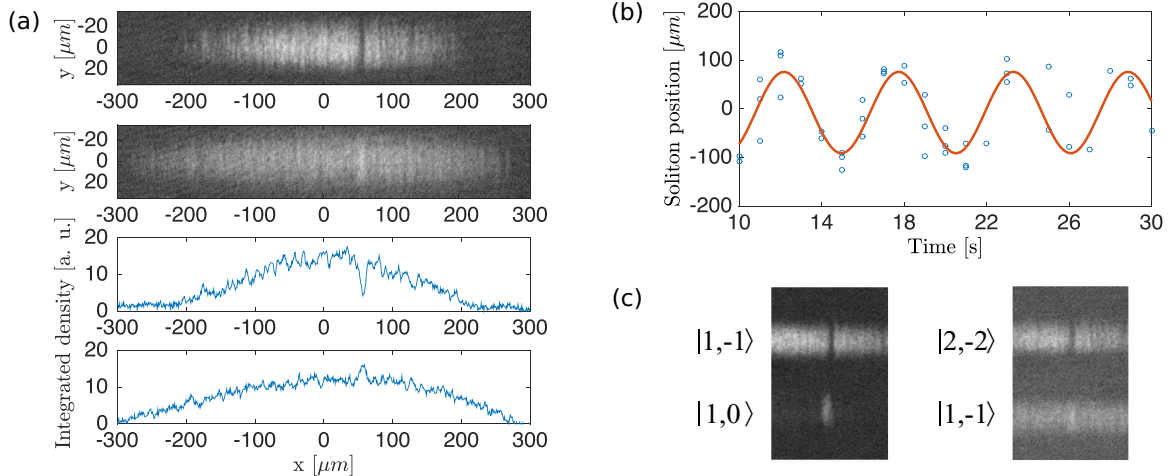


FIG. 1. Experimental realization of dark-antidark solitary waves. (a) Absorption images (upper two panels) and corresponding integrated cross sections (lower two panels) of a dark-antidark solitary wave. The dark soliton component resides in a cloud of $|F, m_F\rangle = |2, -2\rangle$ atoms (upper and third panel), while the bright component consists of atoms in the $|F, m_F\rangle = |1, -1\rangle$ state (second and fourth panel from top). (b) Experimentally observed oscillation of the dark-antidark solitary wave in the trap. The position is measured along the x axis, i.e., along the weakly confining axis of the trap. The time is measured starting from the initial microwave pulse that creates the two-component mixture. The blue dots are experimental data, while the red line is a sine function fit to the data. (c) Comparison between a DB soliton in a mixture of atoms in the $|F, m_F\rangle = |1, -1\rangle$ and $|F, m_F\rangle = |1, 0\rangle$ states (left image) and a dark-antidark structure in a mixture of atoms in the $|F, m_F\rangle = |2, -2\rangle$ and $|F, m_F\rangle = |1, -1\rangle$ states (right image).

atoms held in an optical trap with harmonic trap frequencies of $\omega_{x,y,z} = 2\pi\{1.4, 176, 174\}$ Hz, where z is the direction of gravity. Evaporation in the dipole trap is continued until no thermal fraction is discernible. Initially, all atoms are in the $|F, m_F\rangle = |1, -1\rangle$ hyperfine state. Subsequently, a homogeneous magnetic bias field of 10 G is turned on, and a brief microwave pulse transfers a fraction of the atoms (approximately 50% for the case shown in Fig. 1) into the $|2, -2\rangle$ hyperfine state. The transfer occurs uniformly across the whole BEC. The resulting two-component BEC can be described by two separate Gross-Pitaevskii equations that are only coupled by the intercomponent scattering length (see the theory section below). The intra- and intercomponent scattering lengths for the two states are $a_{11} = 100.4a_0$, $a_{22} = 98.98a_0$, and $a_{12} = 98.98a_0$, where a_{11} denotes the scattering of two atoms in the $|1, -1\rangle$ state, a_{12} the scattering between an atom in the $|1, -1\rangle$ and $|2, -2\rangle$ state, and a_{22} between two atoms in the $|2, -2\rangle$ state [30]; a_0 is the Bohr radius. Based on the standard miscibility argument [26–28], discussed above, this mixture is slightly miscible. After the microwave pulse, the difference between a_{11} and a_{22} induces dynamics leading to a slight enhancement of the concentration of $|2, -2\rangle$ atoms towards the center of the cloud. These dynamics also result in the emergence of a dark-antidark solitary wave as shown in Fig. 1 when the mixture is held in trap for approximately 10 sec or longer. The solitary waves are imaged by suddenly switching off the trap and imaging the $|2, -2\rangle$ state after 10 ms of expansion and the $|1, -1\rangle$ state after 11 ms of expansion. The difference in the free-fall time separates the two images on the camera, so that the two components appear below each other in the images. During all in-trap evolution leading to the soliton formation, the two components have been well overlapped vertically. Repeating this procedure with well-controlled experimental parameters, we observe that each iteration of the experimental run reliably produces a two-component BEC containing one dark-antidark solitary wave such as the one shown in Fig. 1. In all iterations, the dark soliton resides in the $|2, -2\rangle$ component, and the antidark soliton is found in the $|1, -1\rangle$ component. The solitary waves are very long lived. We have observed their in-trap dynamics for up to 20 sec from the time of their generation. For comparison, starting with a 50/50 mixture of the two components, we measure the $1/e$ lifetime of the $|2, -2\rangle$ component to be ~ 22 sec and that of $|1, -1\rangle$ component to be ~ 33 sec for our experimental parameters. The emergence of these solitary waves is rather insensitive to the exact mixture ratio of the components. Experimentally we tested and confirmed their existence in a variety of mixtures ranging from 30% of the atoms in the $|2, -2\rangle$ state and 70% in the $|1, -1\rangle$ state, to mixtures of 50% in the $|2, -2\rangle$ state and 50% in the $|1, -1\rangle$ state. In mixtures where the abundance of the $|2, -2\rangle$ component exceeded approximately 50%, no clear soliton formation was seen. We note that these observations regarding the emergence of the dark-antidark solitons may be specific to the procedure employed to generate these features in our experiments, which is not based on a dedicated wave function engineering [14,17] or phase imprinting [13] technique.

We have also repeated the experiment with a mixture of atoms in the $|1, -1\rangle$ and $|1, 0\rangle$ components. The scattering lengths for this mixture are $a_{11} = 100.4a_0$, $a_{22} = 100.86a_0$,

and $a_{12} = 100.41a_0$, where a_{11} now denotes the scattering of two atoms in the $|1, -1\rangle$ state, a_{12} the scattering between an atom in the $|1, -1\rangle$ and $|1, 0\rangle$ state, and a_{22} between two atoms in the $|1, 0\rangle$ state [30]. This mixture is closer to the miscibility-immiscibility threshold. Following an analogous procedure as described above, no formation of dark-antidark solitary waves was observed. Instead, DB solitons were generated. A comparison between a DB soliton and a dark-antidark one shown in Fig. 1(c), showcasing their very different structure in the bright component. This emphasizes the important role that the miscibility of the component plays for the generation of a no-zero background in the second (bright) component, as has also been highlighted in Ref. [20].

We note here an important open issue that we will not fully address in our numerical investigation that follows, namely that of experimentally identifying only dark-antidark states in the first of the above mixtures and only DB ones in the second one. Our numerical results below will show that dark-antidark states exist up to the miscibility-immiscibility threshold (although as the latter limit is approached, they may be difficult to identify given their degeneration to dark solitons). On the other hand, the work in Ref. [31] suggests that dark-bright solitary waves exist on both sides of the miscibility-immiscibility transition. This combination indicates that a closer investigation of the proximity to the transition limit [including a consideration of possible small residual magnetic field curvatures and genuinely three-dimensional (3D) computations] may be necessary to further explore this feature.

The long lifetimes and reproducible generation of dark-antidark solitary waves in a mixture of atoms in the $|2, -2\rangle$ and $|1, -1\rangle$ states allow us to observe their in-trap dynamics [Fig. 1(b)]. We clearly detect a slow oscillation of the solitary wave along the weak axis of the trap. A fit of the data in Fig. 1(b) gives an oscillation period of approximately 5.6 sec. For comparison, the period of a dark soliton in a single-component BEC in the same trap is predicted to be 1 sec [32,33]. Hence, the dark-antidark solitary waves are significantly slower. A similar trend has been found in Refs. [12,15], where it was seen that dark-bright solitons are slowed when the amount of atoms in the bright component is increased. The theory of “magnetic solitons” described in Ref. [20] assumes $a_{11} = a_{22}$, which in our experiment is only approximately fulfilled. This theory predicts an oscillation period on the order of 9.8 sec, somewhat longer than that observed in the experiment. A quantitative comparison between experiment and theory, including the possible influence of a small residual magnetic field curvature, the mixing ratio of the two components, and the finite lifetime of the trapped atoms, will be left for future work. Here, these first observations of dark-antidark solitary waves serve as a motivation to investigate (chiefly as a function of the intercomponent scattering length) and generalize, for instance, to higher dimensions, the underlying concepts of dark-antidark structures using numerical continuation studies and Bogoliubov–de Gennes analysis.

III. THEORETICAL AND NUMERICAL RESULTS

In order to capture the qualitative features of the states of interest, it will suffice to utilize a mean-field model in the form of the Gross-Pitaevskii equation. Upon suitable

standard reductions [1,2], the model can be transformed to its dimensionless version in the form

$$i \frac{\partial \psi_1}{\partial t} = -\frac{1}{2} \Delta \psi_1 + V(\mathbf{x}) \psi_1 + (g_{11} |\psi_1|^2 + g_{12} |\psi_2|^2) \psi_1, \quad (1)$$

$$i \frac{\partial \psi_2}{\partial t} = -\frac{1}{2} \Delta \psi_2 + V(\mathbf{x}) \psi_2 + (g_{12} |\psi_1|^2 + g_{22} |\psi_2|^2) \psi_2. \quad (2)$$

Here, the pseudo-spinor field is denoted by $(\psi_1, \psi_2)^T$ (where T is used for transpose), $V(\mathbf{x}) = \frac{\Omega^2}{2} \mathbf{x}^2$ represents the parabolic trap, while g_{ij} are the interaction coefficients, proportional to the experimental scattering lengths mentioned above. In line with the analysis of Ref. [20], we will assume in what follows that $g_{11} = g_{22} = g$, while $0 \leq g_{12} \leq g$. Given that only the ratios of the scattering lengths matter, we will choose $g = 1$, while $0 \leq g_{12} < 1$, in order to be in the miscible regime, while preserving intercomponent repulsion.

In our numerics, the stationary states $(\psi_1^{(0)}, \psi_2^{(0)})^T$ are identified by virtue of a fixed point iteration (typically a Newton method) originally at $g_{12} = 0$, i.e., the limit where the two components are uncoupled. Then, parametric continuation is utilized in order to follow the configuration as a function of g_{12} up to the miscibility threshold.

Upon computing the solution, Bogoliubov–de Gennes stability analysis is implemented that perturbs the solutions according to

$$\psi_1 = e^{-i\mu_1 t} [\psi_1^{(0)}(\mathbf{x}) + \delta(a(\mathbf{x})e^{i\omega t} + b^*(\mathbf{x})e^{-i\omega^* t})], \quad (3)$$

$$\psi_2 = e^{-i\mu_2 t} [\psi_2^{(0)}(\mathbf{x}) + \delta(c(\mathbf{x})e^{i\omega t} + d^*(\mathbf{x})e^{-i\omega^* t})]. \quad (4)$$

Here ω represents the linearization eigenfrequency, and the vector $(a, b, c, d)^T$ is the linearized eigenvector pertaining to the respective eigenfrequency. The chemical potentials are denoted by (μ_1, μ_2) , while δ is a formal small parameter of the linearization ansatz. Ω represents the strength of the trapping potential in the longitudinal versus the transverse directions (i.e., the ratio thereof). For 1D calculations, one needs to choose $\Omega \ll 1$ for the reductions to the 1D effective model to be meaningful: we set $\Omega = 0.025$ and $\mu_1 = \mu_2 = 1$. For 2D calculations, we will assume $\Omega = 0.2$ and $\mu_1 = \mu_2 = 2$ unless noted otherwise.

Numerical computations are performed using FreeFem++ [34]. The numerical system developed for computing stationary solutions of the Gross-Pitaevskii equation [35] was extended for the two-component system (1)–(2). We use quadratic (P^2) finite elements with mesh adaptivity, offering high resolution of the steep gradients in the solution. The Bogoliubov–de Gennes linear eigenvalue problem corresponding to the P^2 finite element discretization is solved using the ARPACK library.

A. 1D: Dark-antidark solitary waves

We start by considering the scenario of dark-antidark solitons in one spatial dimension. Solutions of this type are represented in Fig. 2(a) for the in-trap case ($\Omega = 0.025$) and Fig. 2(b) for the homogeneous case ($\Omega = 0$). The latter case

(without trap) is shown for reference to classical dark-bright solitons. In the latter, contrary to what is the case here (where it is finite), the background of this second component is vanishing. In what follows, we focus on the case with trapping potential.

For $g_{12} = 0$ the solution has the form of a regular dark solitary wave coupled to a ground state in the second component. For a finite value of g_{12} the stationary solution develops a bump at the location of the dark soliton dip. Due to the intercomponent repulsion, the density dip in the first component leads to an attracting potential well for the second component.

Therefore an antidark peak is formed that becomes stronger as g_{12} is increased, while the first component tends to vanish as the miscibility-immiscibility threshold of $g_{12} = 1$ is approached. This trend is clearly seen in Fig. 2(a). For this case, we have used $\Omega = 0.025$, although similar results have been found for other values of the trap strength.

Remarkably, in the case of the dark-antidark solitary wave family we find the relevant solution to be generally stable [as shown in Fig. 2(c)] through a wide interval of variation of the g_{12} parameter; an extremely weak oscillatory instability arises for $0.71 < g_{12} < 0.87$, which will be discussed further below, yet its growth rate is so small that we do not expect it to affect the dynamics in an observable way over the time scales of interest. In the large chemical potential limit, we in fact have a detailed handle on the spectrum of the relevant eigenfrequencies in an analytical form. When $g_{12} = 0$, the second component is uncoupled from the first and its spectrum in the ground state consists of eigenfrequencies

$$\omega = \sqrt{\frac{n(n+1)}{2}} \Omega, \quad (5)$$

where n is a non-negative integer, as discussed in Refs. [36,37]. The dark soliton (DS) spectrum on the other hand, as explained, e.g., in Ref. [2], consists of the spectrum of the ground state in which the soliton is “embedded,” given by Eq. (5), as well as an extensively studied anomalous (or negative energy) mode associated structurally with the excited state nature of the DS state, and practically with its oscillation inside the parabolic trap. As is well known, the latter mode has the frequency $\omega = \Omega/\sqrt{2}$ in the large chemical potential limit [32,33] and is the lowest excitation frequency in the system. Using an argument similar to that presented in Ref. [20], we expect that the relevant mode scales as $\omega \approx \Omega \sqrt{\delta g / (2g)}$, where $\delta g = g - g_{12}$. This prediction is also represented in Fig. 2(c) (black dots in the plot of the real part of eigenvalues) and is in reasonable agreement with the corresponding numerical result throughout the interval of variation of g_{12} . As $\Omega \rightarrow 0$ this mode tends to 0, restoring the translational invariance in the limit when the trap is absent. In the presence of the trap, there are two pairs of modes at $\omega = 0$ associated with the invariance of both components with respect to phase and physically associated with the conservation of atom numbers in both of the nonlinearly coupled components. Regarding the rest of the spectrum (the modes associated with the ground state in each component), it can be seen that they can be partitioned into two fundamental categories, namely those that are essentially left invariant and those that undergo a rapid

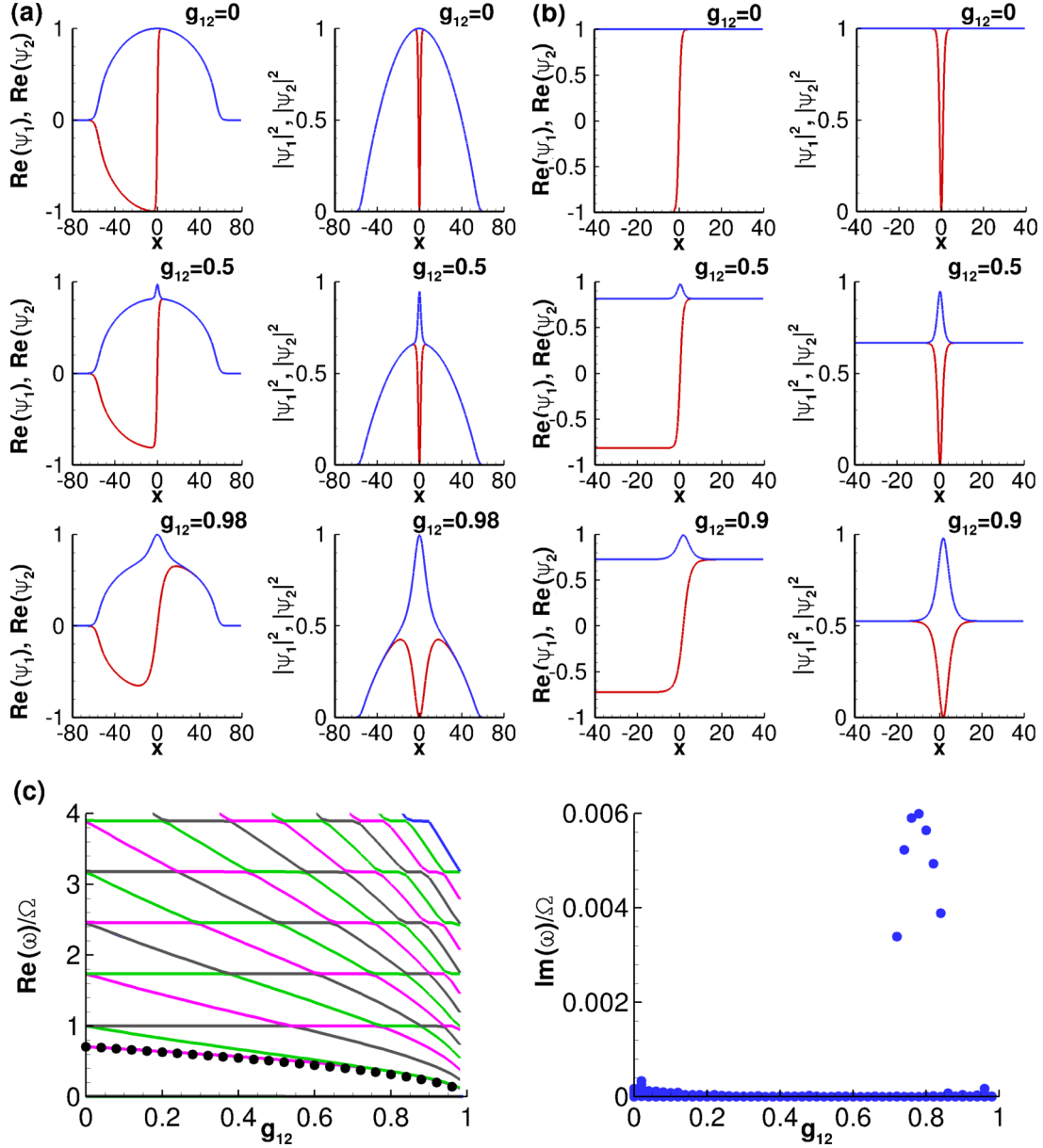


FIG. 2. 1D Dark-antidark solitary waves. Three examples of the two components in the dark-antidark state for progressively increasing g_{12} : (a) in-trap case with trap frequency $\Omega = 0.025 \ll 1$, (b) homogeneous case with $\Omega = 0$. (c) Dependence of the lowest eigenfrequencies scaled by the trap frequency Ω for the in-trap case. Plot of the real part of eigenvalues: black dots indicate the theoretical prediction for the anomalous mode (see discussion in the text). The colors are there only to visually aid the eye to identify the continuation of the different modes. Plot of the imaginary part: evidence of very low growth of oscillatory instability for $0.71 < g_{12} < 0.87$.

monotonic decrease (which is nearly linear for small g_{12}) as g_{12} increases. The former ones of these modes are referred as density modes in Ref. [1], while the latter ones [decreasing proportionally to $\sqrt{(g - g_{12})/(g + g_{12})}$] are referred to as spin modes. As this takes place, it is in principle possible for the anomalous mode (of dark soliton oscillation) and the lowest frequency associated with the ground state to collide and lead to a resonant eigenfrequency quartet [2]. This does happen in the example of Fig. 2(c) (plot of the imaginary part of eigenvalues) for $0.71 < g_{12} < 0.87$, yet as mentioned above the growth rate of this instability is minuscule, and hence it will not be considered further herein.

B. 2D: Vortex-antidark solitary waves

We now generalize the above concept to the 2D case, as is shown in Fig. 3. Our first example in this setting replaces the 1D dark soliton by a 2D vortex in the first component that generates an attractive potential well for the second component. Once again, it can be seen that starting from the decoupled limit of $g_{12} = 0$ and increasing g_{12} in the miscible regime, the coupled vortex-antidark state emerges with progressively more and more atoms of the second component being radially concentrated in the well formed by the vortex. This creates a radially symmetric antidark solitary wave in the second component.

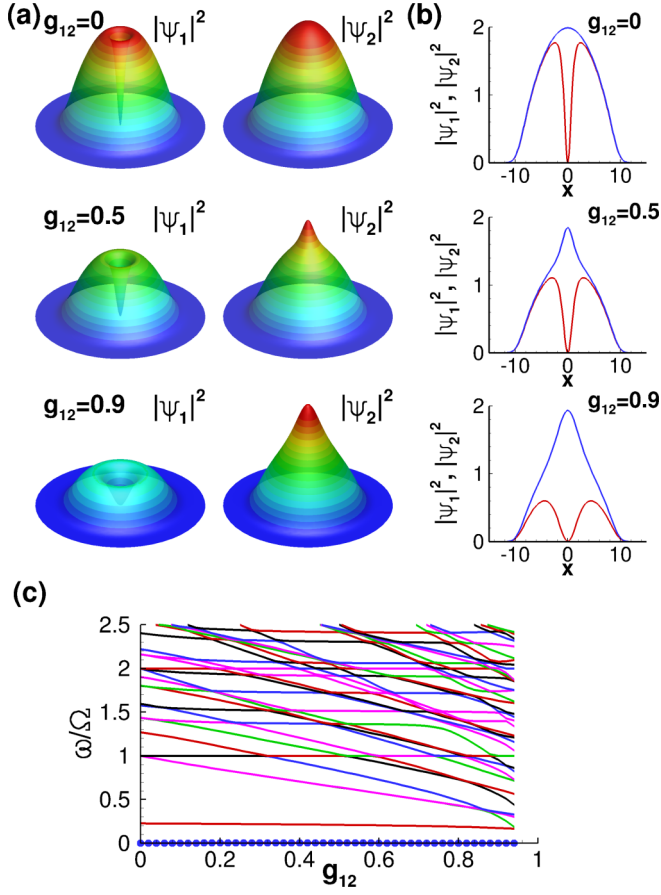


FIG. 3. 2D Vortex-antidark solitary waves obtained for $\Omega = 0.2$ and $\mu_1 = \mu_2 = 2$. (a) Three-dimensional renderings of the density as a function of (x, y) for the vortex-antidark states and for the particular values of $g_{12} = 0, 0.5$, and 0.9 . (b) Cut through the density of these states (at $y = 0$). (c) Variation of eigenfrequencies normalized by the trap frequency Ω of the spectral BdG analysis with g_{12} (see also the relevant detailed discussion in the text). Notice that all eigenfrequencies remain real over the parametric interval considered, indicating the spectral stability and dynamical robustness of the associated state.

For this case, the spectrum, shown in Fig. 3(c) features two sets of constituents. In the limit of $g_{12} = 0$, the ground state component consists of frequencies theoretically approximated (in the Thomas-Fermi limit of large μ considered here) by

$$\omega = \sqrt{m + 2k(1 + m) + 2k^2\Omega}, \quad (6)$$

where $k, m \geq 0$ are non-negative integers. The first component bearing the vortex also shares these frequencies (due to the vortex being again “embedded” within the ground state), but additionally carries an anomalous (negative energy) mode that is a signature of its excited state nature [2]. This mode is associated with the rotation of the vortex around the center of the parabolic trap. It is well characterized by the frequency $\omega = \frac{\Omega^2}{2\mu} \log(A\mu/\Omega)$, where $A \approx 2\sqrt{2}\pi$ [38]. As g_{12} is increased, a similar trend as in one dimension is observed: one set of frequencies remains invariant, while a second set, originally degenerate with the first at $g_{12} = 0$, monotonically decreases

as $\delta g \rightarrow 0$. The anomalous mode frequency associated with the vortex monotonically decreases as well, although in a less pronounced manner.

Physically, this implies that the additional atoms stored within the antidark component decrease the rotation frequency (i.e., increase the rotation period) of the composite entity within the trap. A similar behavior was found for dark-bright solitons even experimentally [12, 15]. Thus, for dark-antidark solitons and for vortex-antidark solitary waves, “heavier means slower.”

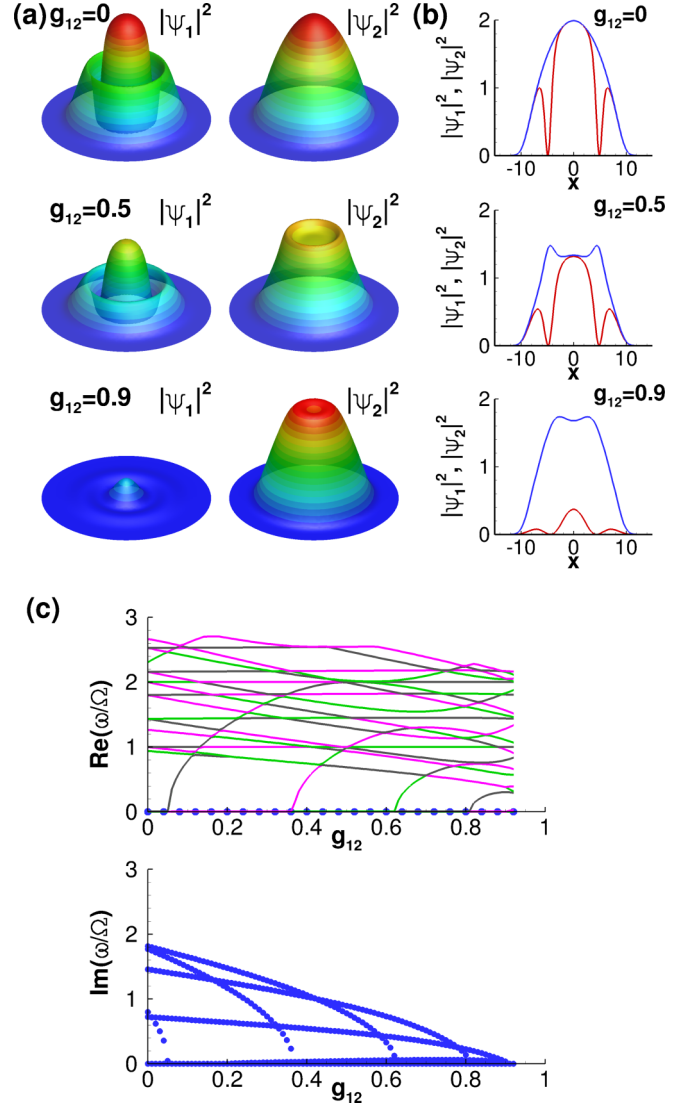


FIG. 4. 2D Ring-antidark-ring solitary waves obtained for $\Omega = 0.2$ and $\mu_1 = \mu_2 = 2$. (a) Three-dimensional renderings of the density as a function of (x, y) for the vortex-antidark states and for the particular values of $g_{12} = 0, 0.5$ and 0.9 . (b) Cut through the density of these states (at $y = 0$). (c) Variation of real and imaginary parts (normalized by the trap frequency Ω) of frequencies of the spectral BdG analysis with g_{12} . Notice how the imaginary eigenfrequencies associated with instabilities to (azimuthal) snaking progressively disappear as g_{12} is increased, i.e., the ring antidark solitary wave has a stabilizing effect on the unstable ring dark soliton.

C. 2D: Ring-antidark-ring solitary waves

Finally, to illustrate the generality of the underlying concept, we explore the role of increasing the intercomponent (repulsive) coupling g_{12} in the miscible regime for the case where the first component bears a ring dark soliton (RDS), while the second one is started in the ground state, again at large μ (i.e., in the vicinity of the Thomas-Fermi limit). Ring dark solitons have been predicted early on in the context of BECs [39], following their proposal and even experimental observation in nonlinear optics [40,41]. Subsequent detailed studies of their stability [42,43] illustrated that they are unstable for all values of the chemical potential from the linear limit onwards, progressively becoming more unstable as μ is increased. The initial instability is towards a quadrupolar mode leading to four vortices, while subsequently hexapolar (leading to six vortices), octapolar (leading to eight vortices), etc., instabilities arise in the relevant spectrum.

Here, we observe the relevant states in Fig. 4. A progressive increase of g_{12} generates an attractive annular potential for atoms in the second component so that an antidark ring is formed in the second component. As with the previous states, the increase of g_{12} also decreases the density of the dark ring component (for the same chemical potential), finally leading it towards extinction as the miscibility-immiscibility threshold is approached. An associated and quite interesting feature is that as g_{12} is increased, progressively more and more unstable modes of the original single component RDS are *stabilized*, hence it is intriguing to note that for the same chemical potential states with larger g_{12} and hence larger antidark component are less susceptible to instability. In fact, the state tends towards complete stabilization as the miscibility threshold is approached, a feature interrelated with the tendency towards extinction of the associated RDS component. It is relevant to note also that in these last two cases (of Figs. 3 and 4) the state appears to terminate *noticeably before* the homogeneous miscibility limit of $g_{12} = 1$, while the 1D corresponding DA state can be identified continuously up to the miscibility-immiscibility threshold.

IV. CONCLUSIONS AND FUTURE WORK

In the present paper, we proposed a multitude of states motivated by the recent proposal of magnetic solitons put forth in Ref. [20]. These “dark-antidark” states that we discuss are based on a simple physical principle, namely, the formation of an attractive potential well on top of a ground state component by the presence of a “dark” structure in the other component. In the miscible case with intercomponent repulsion, this potential well attracts atoms and forms an antidark entity (a 1D soliton, a 2D soliton, or even a ring soliton) in the (formerly) ground state component. This is a natural generalization (in the case of

a nonvanishing background for the second component) also of the notion of DB solitons which has recently been extensively explored.

We have proposed this notion theoretically at an intuitive level and have illustrated its generality via detailed numerical computations. Furthermore, we have showcased it in experiments, at least in as far as its 1D installment is concerned. This emphasizes the relevance of these features for current experiments with multicomponent BECs. An additional appealing feature of such symbiotic structures lies in the fact that when the single-component entity (such as the ring dark soliton) may be unstable, this coupling appears to have a stabilizing effect rendering the relevant entity more amenable to observation. Both the existence and the spectral stability characteristics of these states were explained over variations of the intercomponent coupling throughout their range of existence.

Naturally, there are many open directions for future study in this subject. From an experimental perspective, it would be particularly interesting to explore the possibility to form such states in both two and three dimensions. In the latter (3D) setting, computations would also be especially useful in elucidating such states: recently, vortex-line-bright and vortex-ring-bright [44] states have been identified, and their generalization to antidark ones would be quite relevant, as well as the study of their stability. From a theoretical perspective, it would also be quite intriguing to explore further the “particle description” of such entities, both at the level of the single particle (e.g., characterizing the rotation of a vortex-antidark solitary wave) but also quite importantly at the level of interaction of multiple such entities. The latter has not been quantified even in 1D settings. As a final but important point, we note the sensitive dependence of the emergence in experiments of the dark-antidark versus DB solitons on the value of scattering lengths in the vicinity of the miscibility threshold. This is a feature worth further elucidating in future experiments and corresponding theoretical analyses. Some of these topics are presently under consideration and will be reported in future publications.

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